SUPPLEMENTAL MODULES (LINEAR ALGEBRA)

LibreTexts™

CHAPTER OVERVIEW

Supplemental Modules (Linear Algebra)

Matrices

2: Determinants and Inverses

3: Linear Systems

4: Linear Programming

Supplemental Modules (Linear Algebra) is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

Matrices

1. Definition of a Matrix

Definition: Matrix

An m by n *matrix* is an array of numbers with m rows and n columns.

Example 1

$$\begin{pmatrix} 4 & 5 \\ 0 & 15 \\ -9 & 3 \end{pmatrix}$$

is a 3 by 2 matrix.

Example 2

Consider the system of equations

$$\begin{aligned} 2x \quad -y \quad +3z &= 5 \quad (1) \\ x \quad\quad\; +4z &= 3 \quad (2) \\ 5x \quad -7y \quad +3z &= 7 \quad (3) \end{aligned}$$

Then the matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 5 \\ 1 & 0 & 4 & 3 \\ 5 & -7 & 3 & 7 \end{array}\right)$$

is called the *augmented matrix* associated to the system of equations. Two matrices are called *equal* if all of their entries are the same. Two matrices are called *row equivalent* is one can be transformed using a sequence of the three operations that we discussed earlier.

1. Interchanging two rows.
2. Multiplying a row by a nonzero constant.
3. Replacing a row with the row + a constant multiple of another row.

2. Solving Linear Systems Using Matrices

We can solve a linear system by writing down its augmented matrix and performing the row operations that we did last time.

Example 3

Solve

$$\begin{aligned} 2x \quad -y \quad +z &= 3 \quad (4) \\ x \quad +y \quad +z &= 2 \quad (5) \\ y \quad -z &= -1 \quad (6) \end{aligned}$$

Solution

We write the associated augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

Now begin solving by performing row operations:

$$R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

$$R_1 \leftrightarrow R-2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

$$R_2 \leftrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -1 & -1 \end{array}\right)$$

$$R_1 - R_2 \rightarrow R_1, \quad R_3 + 3R_2 \rightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -1 & -1 \end{array}\right)$$

$$R_1 - R_2 \rightarrow R_1, \quad R_3 + 3R_2 \rightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -4 & -4 \end{array}\right)$$

$$R_3 \rightarrow -\frac{1}{4}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$R_1 - 2R_3 \rightarrow R_1, \quad R_2 + R_3 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$R_1 - 2R_3 - \rightarrow + R_1, \quad R_2 + R_3 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

We can now put the matrix back in equation form:

$$x = 1, y = 0 \text{ and } z = 1$$

Note

If we had seen a bottom row that was of the form $0\ 0\ 0\ a$ where a is a nonzero constant, then there would be no solution. If a had been 0 there would be infinitely many solutions.

3. Addition and Scalar Multiplication of Matrices

We can only add matrices that are of the same dimensions, that is if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 5 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 \\ 7 & 2 \end{pmatrix}$$

then only $A+C$ makes sense. We write

$$A+C=\begin{pmatrix}1+1 & 2+3\\ 3+7 & 4+2\end{pmatrix}=\begin{pmatrix}2 & 5\\ 10 & 6\end{pmatrix}$$

For any matrix, we can multiply a matrix by a real number as in the following example (Same B as above):

$$5B=\begin{pmatrix}10 & 15\\ 20 & 5\\ 25 & 45\end{pmatrix}$$

We define the *zero matrix* to be the matrix with only zeros for entries. For example, the 2 by 2 zero matrix is

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

4. Multiplication of Matrices

To multiply matrices, unfortunately the definition is not the obvious one. We can only multiply matrices where the number of columns of the first matrix is the same as the number of rows of the second matrix. The best way to learn how to multiply matrices is by example:

$$\text{Let } A=\begin{pmatrix}3 & 5 & 2\\ 0 & 1 & -2\end{pmatrix},\ \text{and } B=\begin{pmatrix}7 & -3\\ -2 & 1\\ 0 & 5\end{pmatrix}$$

$$\text{then } AB=\begin{pmatrix}3(7)+4(-2)+2(0) & 3(-3)+4(1)+2(5)\\ 0(7)+1(-2)+-2(5) & 0(-3)+1(1)+-2(5)\end{pmatrix}=\begin{pmatrix}13 & 5\\ -12 & -9\end{pmatrix}$$

Exercise

$$\text{Let } A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\quad B=\begin{pmatrix}4 & 2 & 1\\ -2 & 0 & 0\\ 1 & 6 & -1\end{pmatrix},\quad C=\begin{pmatrix}1 & 0\\ 2 & 1\\ 4 & 5\end{pmatrix},$$
$$D=\begin{pmatrix}3 & 4 & 0\\ 5 & 0 & 0\end{pmatrix},\quad B=\begin{pmatrix}3 & 4 & 2\\ 1 & 5 & 0\\ 1 & -1 & 2\end{pmatrix}$$

Evaluate each one that makes sense:

1) $A+B$ 2) $4C$ 3) AB 4) CD 5) DC 6) $B+E$ 7) A^3

5. Applications of Matrices

Application 1

A) Tables and chairs are made in the Mexico plant, the Brazil plant, and the US plant. The matrix below represents the quantity made per day.

A :

Quantity

	Mexico	Brazil	US
Tables	15	10	50
Chairs	30	12	75

Labor and material cost for 1997 are represented in the following matrix.

$B =$

	Labor	Material
Mexico	15	20
Brazil	12	10
US	30	5

In 1997, the costs have increases to

$C =$

	Labor	Material
Mexico	17	25
Brazil	15	15
US	45	10

Find the following and describe what they mean:

1) AB 2) $C - B$ 3) AC 4) $A(C - B)$ 5) $365AC$

Application 2

Suppose that you have two jobs, each contribute to two different mutual funds for retirement. The first fund pays 5% interest and the second pays 8% interest. Initially $5,000 is put into the funds and after one year there will be $5,300. If the first fund got half of the money from the first job and one third of the money from the second job, how much did each job contribute?

Hint: Multiplication of matrices is the same as composition of functions

Larry Green (Lake Tahoe Community College)

- Integrated by Justin Marshall.

This page titled Matrices is shared under a not declared license and was authored, remixed, and/or curated by Larry Green.

2: Determinants and Inverses

Determinants

Consider row reducing the standard 2x2 matrix. Suppose that a is nonzero.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{1}{a}R_1 \to R_1, \quad R_2 - cR_1 \to R_2$$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{cb}{a} \end{pmatrix}$$

Now notice that we cannot make the lower right corner a 1 if

$$d - \frac{cb}{a} = 0$$

or

$$ad - bc = 0.$$

Definition: The Determinant

We call $ad - bc$ the *determinant* of the 2 by 2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

it tells us when it is possible to row reduce the matrix and find a solution to the linear system.

Example 2.1

The determinant of the matrix

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

is

$$3(2) - 1(5) = 6 - 5 = 1.$$

Determinants of 3 x 3 Matrices

We define the determinant of a triangular matrix

$$\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}$$

by

$$\det = abc.$$

Notice that if we multiply a row by a constant k then the new determinant is k times the old one. We list the effect of all three row operations below.

Theorem

The effect of the the three basic row operations on the determinant are as follows

1. Multiplication of a row by a constant multiplies the determinant by that constant.
2. Switching two rows changes the sign of the determinant.
3. Replacing one row by that row + a multiply of another row has no effect on the determinant.

To find the determinant of a matrix we use the operations to make the matrix triangular and then work backwards.

Example 2.2

Find the determinant of

$$\begin{pmatrix} 2 & 6 & 10 \\ 2 & 4 & -3 \\ 0 & 4 & 2 \end{pmatrix}$$

We use row operations until the matrix is triangular.

$$\frac{1}{2}R_1 \to R_1 (\text{Multiplies the determinant by } \frac{1}{2})$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & -3 \\ 0 & 4 & 2 \end{pmatrix}$$

$$R_2 - 2R_1 \to R_2 \text{ (No effect on the determinant)}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -13 \\ 0 & 4 & 2 \end{pmatrix}$$

Note that we do not need to zero out the upper middle number. We only need to zero out the bottom left numbers.

$$R_3 + 2R_2 \to R_3 \text{ (No effect on the determinant).}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -13 \\ 0 & 0 & -24 \end{pmatrix}$$

Note that we do not need to make the middle number a 1.

The determinant of this matrix is 48. Since this matrix has $\frac{1}{2}$ the determinant of the original matrix, the determinant of the original matrix has

$$\text{determinant} = 48(2) = 96.$$

Inverses

We call the square matrix *I* with all 1's down the diagonal and zeros everywhere else the *identity* matrix. It has the unique property that if A is a square matrix with the same dimensions then

$$AI = IA = A.$$

Definition

If A is a square matrix then the inverse A^{-1} of A is the unique matrix such that

$$AA^{-1} = A^{-1}A = I.$$

Example 2.3

Let

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

then

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

Verify this!

Theorem: ExistEnce

The inverse of a matrix exists if and only if the determinant is nonzero.

To find the inverse of a matrix, we write a new extended matrix with the identity on the right. Then we completely row reduce, the resulting matrix on the right will be the inverse matrix.

Example 2.4

$$\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$$

First note that the determinant of this matrix is

$$-2 + 1 = -1$$

hence the inverse exists. Now we set the augmented matrix as

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right)$$

$$R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

$$R_2 - 2R_1 \rightarrow R_2$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

$$R_1 + R_2 \rightarrow R_1$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

Notice that the left hand part is now the identity. The right hand side is the inverse. Hence

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

Solving Equations Using Matrices

Example 2.5

Suppose we have the system

$$2x - y = 3$$

$$x - y = 4$$

Then we can write this in matrix form

$$Ax = b$$

where

$$A = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and } b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

We can multiply both sides by A^{-1}:

$$A^{-1}Ax = A^{-1}b$$

or

$$x = A^{-1}b$$

From before,

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

Hence our solution is

$$\begin{pmatrix} -1 & -5 \end{pmatrix}$$

or

$$x = -1 \text{ and } y = 5$$

Contributors

- Larry Green (Lake Tahoe Community College)
- Integrated by Justin Marshall.

This page titled 2: Determinants and Inverses is shared under a not declared license and was authored, remixed, and/or curated by Larry Green.

3: Linear Systems

1. Geometry of Systems of Equations

We know that for two by two linear systems of equation, the geometry is that of two lines that either intersect, are parallel, or are the same line. If they intersect then there is exactly one solution, if they are parallel then there are no solutions, and if they are the same line, then there are infinitely many solutions. For three by three systems, the situation is different. The solution set is either the empty set, a point, a line, or a whole plane. For four by four systems, the geometry becomes four dimensional and is rough to comprehend, but is still useful.

2. The Algebra of Linear Systems

In this class we will perform algebra on linear systems in a new way. For example, for a three by three system, we line the equations up to form three rows. We will manipulate the rows to simplify the equations. There are three operations, called row operations that we can perform:

$$cR_j + R_i \to R_i \tag{3.1}$$

We follow the following steps that use row operations to solve a system of equations.

1. We interchange rows so that the top left corner is nonzero (preferably a 1).
2. We multiply row 1 by the appropriate number to make the top left corner a 1.
3. We replace rows two and three with the appropriate multiples of row 1 + rows two and three to make the two bottom rows begin with 0.
4. We interchange the two bottom rows so that the middle coefficient nonzero.
5. We multiply row two appropriately so that the middle number is a 1.
6. We use row two to make the middle top and bottom zero.
7. Make the bottom right 0.
8. Use row three to make the top and middle rights 0.

$$\begin{aligned} x &= 1 && (3.2)\\ y &= 2 && (3.3)\\ z &= 3 && (3.4) \end{aligned}$$

3. Exercises

$$\begin{aligned} 4x - 3y - z &= 0 && (3.5)\\ x - 3y + 2z &= 7 && (3.6)\\ 3x + 9y - z &= -2 && (3.7) \end{aligned}$$

$$\begin{aligned} 3x - 2y + z &= 3 && (3.8)\\ 4x + y &= 1 && (3.9)\\ 11y - 4z &= -9 && (3.10) \end{aligned}$$

4. **Application**

Your breakfast consists of orange juice, cereal, and eggs with the following nutritional information:

	OJ	Cereal	Eggs
Protein	0%	10%	20%
Vitamin C	20%	15%	0%
Calories	100	120	100

If you must have 30% protein, 30% Vitamin C and 300 calories for your breakfast, How many servings of OJ, Cereal, and Eggs should you have?

Solution

- Let

  $x =$ the number of servings of OJ
  $y =$ the number of servings of Cereal
  $z =$ the number of servings of eggs

  Then

$$\begin{aligned} 10y &+20z = 30 && (3.11)\\ 20x &+15y = 30 && (3.12)\\ 100x &+120y &+100z = 300 && (3.13) \end{aligned}$$

We conclude that the breakfast should consist of 1.5 of a serving of OJ, no cereal, and 1.5 servings of eggs.

1.

$$\begin{aligned} 20x &+15y = 30 && (3.14)\\ 10y &+20z = 30 && (3.15)\\ 100x &+120y &+100z = 300 && (3.16) \end{aligned}$$

2.

$$\begin{aligned} x &+\frac{3}{4}y = \frac{3}{2} && (3.17)\\ 10y &+20z = 30 && (3.18)\\ 100x &+120y &+100z = 300 && (3.19) \end{aligned}$$

3.

$$\begin{aligned} x &+\frac{3}{4}y = \frac{3}{2} && (3.20)\\ 10y &+20z = 30 && (3.21)\\ 45y &+100z = 150 && (3.22) \end{aligned}$$

4. Already Done

5.

$$\begin{aligned} x &+\frac{3}{4}y = \frac{3}{2} && (3.23)\\ y &+2z = 3 && (3.24)\\ 45y &+100z = 150 && (3.25) \end{aligned}$$

6 .

$$\begin{aligned} x &-\frac{3}{2}z = -\frac{3}{4} && (3.26)\\ y &+2z = 3 && (3.27)\\ &10z = 15 && (3.28) \end{aligned}$$

7.

$$x - \frac{3}{2}z = -\frac{3}{4} \tag{3.29}$$
$$y + 2z = 3 \tag{3.30}$$
$$z = \frac{3}{2} \tag{3.31}$$

8.

$$x = \frac{3}{2} \tag{3.32}$$
$$y = 0 \tag{3.33}$$
$$z = \frac{3}{2} \tag{3.34}$$

Larry Green (Lake Tahoe Community College)

- Integrated by Justin Marshall.

3: Linear Systems is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

4: Linear Programming

1. Linear Programming **(An Example)**

Maximize

$$P = 2x + 5 \tag{4.1}$$

subject to the constraints

$x + 3y \leq 15$
$4x + y \leq 16$
$x \geq 0$
$y \geq 0$
First we graph the system of inequalities.

- For
  $x + 3y = 15$
  we use $(0, 5)$ and $(15, 0)$ and note that the arrows point towards $(0, 0)$.
- For
  $4x + y = 16$
  we use $(0, 16)$ and $(4, 0)$ and note that the arrows point towards $(0, 0)$.
- $x \geq 0$ and $y \geq 0$ constrain the region in the first quadrant.

We have a closed and bounded region (closed means that all the boundary lines are solid). We now need to find the largest value that

$$P = 2x + 5y \tag{4.2}$$

gives when (x, y) is inside the region. Lets try some guesses.

1. If $P = 0$ we can graph
   $2x + 5y = 0$
   and note that the graph is a line that intersects the region.
2. If $P = 1$ we can graph
   $2x + 5y = 1$
   and note that the graph is a straight line that intersects the region. Notice also that these lines are parallel.
3. If $P = 10$, we again get a parallel line that intersects the region.
4. If $P = 30$, we get a parallel line that does not intersect the region. Therefore, P must be smaller than 30.

We are searching for the parallel line that is farthest from the origin that intersects the region. this occurs at the corner point $(3, 4)$.

2. The Big Theorem

Theorem: Fundamental Theorem of Linear Programming

If a linear programming problem has a solution, then the solution always occurs at a *corner point*. If two adjacent corner points give solutions, then every point on the line segment connecting them also give that solution. If the profit function is

$$P = ax + by$$

then if the region is bounded, then P obtains both a maximum and a minimum.

1. If a and b are positive, then P will always have a minimum in the region if the region lies entirely within the first quadrant.

Step by Step Process

1. Graph the region and list the corner points in a table. You find the corner points by using matrices. Note if the region is bounded, or if we are looking for a minimum and that region is restricted to the first quadrant.
2. In the second column evaluate the Profit function
3. If we are looking for a maximum, circle the largest number in the profit column. If there is a unique maximum then this is the solution. If there are two maxima, then the entire line segment between the two points gives a solution. If we are looking for a minimum, then circle the smallest number in the profit column, and proceed as before.

$$80x + 50y \qquad (4.3)$$

subject to the constraints
$2x + y \leq 80$
$x + y \leq 50$
$x \geq 0$
$y \geq 0$

We graph the equations and notice that there are three easy to find corner points

$(0, 0), (40, 0), (0, 50)$

The forth corner point is at the intersection of the two lines. The matrix is

$A =$

2	1
1	1

Note that the determinant is

$$2 - 1 = 1$$

so that a solution exists. We use a calculator to find

$A^{-1} =$

1	-1
-1	2

Hence the solution is

$$(30, 20)$$

Now construct a profit table:

Point	$80x + 50y$
$(0, 0)$	0
$(40, 0)$	3,200
$(0, 50)$	2,500
$(30, 20)$	3,400

Since the largest number is 3,400, we can conclude that the maximum occurs at the point $(30, 20)$ with a value of 3,400.

3. Linear Programming Applications

Solomon manufactures parabolic skis and conventional skis. The manufacturing process is done first at the fabricating department and then at the finishing department. The fabricating department has at most 108 labor hours per day available while the finishing department has at most 24 labor hours available per day. The parabolic ski requires 6 labor hours at the fabricating department and one labor hour at the finishing department. The conventional ski requires 4 labor hours at the fabricating department and one labor hour at the finishing department. Solomon makes a $40 profit on its parabolic skis and $30 on its conventional skis. How many of each type of ski should be manufactured each day to realize a maximum profit.

Harveys is planning to add an additional room to its casino. The casino consists of card tables and slot machines. On an average day, a slot machine makes $2,000, while a card table makes $30,000. The total square feet available at Harveys' new casino room for gaming is 3,000 square feet and the total number of additional employees allowed by the gaming commission is 45. If a slot machine occupies 6 square feet and requires one employee per 12 machines and a card table occupies 54 square feet and requires three employees per two tables, how many slot machines and how many card tables should be put into the new casino room in order to maximize profit? What is the maximum Profit? (Smaller tables are permitted)

A veterinarian wishes to prepare a supply of dog food by mixing Purina Big chunks and Alpo Meaties. The mixture is to contain at least 50 units of carbohydrates, at least 36 units of protein, and at least 40 units of fat. The Big Chunks contain 3 units of carbohydrates per pound, 3 units of protein per pound, and 2 units of fat per pound. The Meaties contain 5 units of carbohydrates per pound, 4 units of protein per pound, and 8 units of fat per pound. The veterinarian can purchase the Big Chunks for 44 cents per pound and the Meaties for 80 cents per pound. How many pounds of each dog food should be mixed in order to meet the dietary requirements at the least cost?

4: Linear Programming is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.